Applications of Sigma Notation in Counting and Evaluating the Number of Solutions of Indeterminate Equations Part I–An Application of Sigma Notation in Counting

M. H. Poursaeed
Lorestan University

Applications of Sigma Notation in Counting and Evaluating the Number of Solutions of Indeterminate Equations

By Mohammad H. Poursaeed

Abstract. Since common rules of counting are not easily applied in some cases, we present an efficient method of counting based on the utility of sigma notation. For example, given n cards, numbered consecutively from 1 to n, we devise a way to determine the number of ways that r cards can be selected from the n in such a way that the relative order of the r cards is preserved.

Next, we introduce a method for counting the number of non-negative integer solutions of an indeterminate equation $\sum_{i=1}^{m-1} a_i x_i + a_m (x_m)^k = n$; in which $k, m, n \in \mathbf{N}$ and such that for $i = 1, 2, \ldots, m-1$; $a_i | a_{i+1}$. We also give a solution for the number of solutions of the corresponding inequality. Finally, we introduce a method for approximating the number of non-negative integer solutions of indeterminate equations and inequalities.

Part I - An Application of Sigma Notation in Counting

Introduction

To determine the number of outcomes of an experiment, which is composed of other experiments, one can use the addition and multiplication principles of the so-called "counting rules." Sometimes, we are faced with a different situation. For example, if we wish to select r cards randomly from n cards, numbered consecutively from 1 to n, in such a way that the numbering of the selected cards is not necessarily consecutive, the multiplication principle can not be applied because the selection of first card up to r^{th} card, is carried out dependently. Let's take a simple case into account as an application of the sigma notation. Suppose that we are going to determine the number of ways in which two cards are selected from n cards, numbered from 1 to n, in such a way that the number of

the first card is less than the second. Note that this is equivalent to determining the number of lattice points (x_1, x_2) in the "Cartesian plane" such that $1 \leq x_1 < x_2 \leq n$. The situation is shown in figure 1.

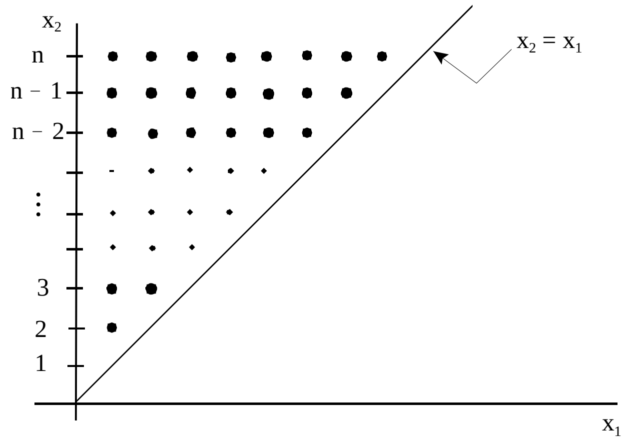

Figure 1 $\{(x_1, x_2) : x_1, x_2 \in \mathbf{Z},\, 1 \leq x_1 < x_2 \leq n\}$

To determine the number of lattice points, by applying sigma notation, we have:

$$\begin{aligned} & |\{(x_1, x_2) : x_1, x_2 \in \mathbf{Z};\ 1 \leq x_1 < x_2 \leq n\}| \\ = & |\{(x_1, x_2) : x_1, x_2 \in \mathbf{Z};\ 1 \leq x_1 < x_2;\ 2 \leq x_2 \leq n\}| \\ = & |\{(x_1, x_2) : x_1, x_2 \in \mathbf{Z};\ 1 \leq x_1 \leq x_2 - 1;\ 2 \leq x_2 \leq n\}| \\ = & \textstyle\sum_{x_2=2}^{n} \sum_{x_1=1}^{x_2-1} 1 = \sum_{x_2=2}^{n} (x_2 - 1) = \frac{n(n-1)}{2}. \end{aligned}$$

Generally, if one experiment A can result in any of m possible outcomes and if another experiment B can result in any of n possible outcomes, and the experiment L is defined by performing experiments A and B in consecutive stages, then L yields $m \cdot n$ outcomes, each of which can be represented as an ordered pair (a, b), where a is an outcome of A and b is an outcome of B. The $m \cdot n$ outcomes can be represented by $m \cdot n$ lattice points inside a rectangle in the Cartesian plane. If there are constraints in the performance of two or more experiments, their corresponding points inside the rectangular or multi-dimensional cell will be considered so that we can easily determine the range of the variables and the bounds of repeated sigma notations in order to determine the number of related lattice points.

Main Results

Based on the aforementioned points, we illustrate the suggested method through one theorem, one problem, and a few examples.

THEOREM 1. *If r cards are selected with replacement from n cards, numbered from 1 to n, and X_i is the number of i^{th} card selected, then:*

$$P\left(X_1 \le X_2 - k_1, X_2 \le X_3 - k_2, \ldots, X_i \le X_{i+1} - k_i, \ldots \right.$$
$$\left. \ldots, X_{r-1} \le X_r - k_{r-1}\right) = \frac{\binom{n+r-1-\sum_{j=1}^{r-1} k_j}{r}}{n^r}.$$

PROOF. In a special case where the k_i 's are assumed to be positive and equal, the problem has been considered (see 1). If the k_i 's are non-negative, then:

$$X_1 \le X_2 - k_1, X_2 \le X_3 - k_2, \ldots, X_i \le X_{i+1} - k_i, \ldots$$
$$\ldots, X_{r-1} \le X_r - k_{r-1}$$

Can be written as follows:

$$1 \le X_1 \le X_2 - k_1 \le X_3 - (k_1 + k_2) \le \ldots \le X_{r-1} - \textstyle\sum_{j=1}^{r-2} k_j$$
$$\le X_r - \textstyle\sum_{j=1}^{r-1} k_j \le n - \sum_{j=1}^{r-1} k_j. \qquad \text{(eq. 1)}$$

We now endeavor to determine the number of lattice points in r-dimensional space, which correspond to inequality 1. Given the fact that $\prod_{i=1}^{n} m_i = \sum_{x_k=1}^{m_k} \sum_{x_{k-1}=1}^{m_{k-1}} \cdots \sum_{x_1=1}^{m_1} 1$ as well as the range of variables in eq. 1, we have:

$$\sum_{x_r=\sum_{j=1}^{r-1} k_j+1}^{n} \; \sum_{x_{r-1}=\sum_{j=1}^{r-2} k_j+1}^{x_r-k_{r-1}} \cdots \sum_{x_2=k_1+1}^{x_3-k_2} \; \sum_{x_1=1}^{x_2-k_1} 1$$

$$= \sum_{x_r=\sum_{j=1}^{r-1} k_j+1}^{n} \; \sum_{x_{r-1}=\sum_{j=1}^{r-2} k_j+1}^{x_r-k_{r-1}} \cdots \sum_{x_3=k_1+k_2+1}^{x_4-k_3} \; \sum_{x_2=k_1+1}^{x_3-k_2} (x_2 - k_1)$$

(eq. 2)

By redefining the index in the right-most sum as $j = x_2 - k_1$, redefining the index in the sum to its immediate left as $j = x_3 - (k_1 + k_2 + 1)$, and by using the fact that $\sum_{j=1}^{n} j = \binom{n+1}{2}$ eq. 2 becomes:

$$\sum_{x_r=\sum_{j=1}^{r-1} k_j+1}^{n} \sum_{x_{r-1}=\sum_{j=1}^{r-2} k_j+1}^{x_r-k_{r-1}} \cdots \sum_{x_3=k_1+k_2+1}^{x_4-k_3} \sum_{j=1}^{x_3-(k_1+k_2)} j$$

$$= \sum_{x_r=\sum_{j=1}^{r-1} k_j+1}^{n} \sum_{x_{r-1}=\sum_{j=1}^{r-2} k_j+1}^{x_r-k_{r-1}} \cdots \sum_{x_4=\sum_{j=1}^{3} k_j+1}^{x_5-k_4} \sum_{j=0}^{x_4-\sum_{i=1}^{3} k_i-1} \binom{j+2}{j}.$$

The previous statement can be re-written as follows (see 2):

$$\sum_{x_r=\sum_{j=1}^{r-1} k_j+1}^{n} \sum_{x_{r-1}=\sum_{j=1}^{r-2} k_j+1}^{x_r-k_{r-1}} \cdots$$

$$\cdots \sum_{x_5=\sum_{j=1}^{4} k_j+1}^{x_6-k_5} \sum_{x_4=\sum_{j=1}^{3} k_j+1}^{x_5-k_4} \binom{x_4-\sum_{j=1}^{3} k_j+2}{3}$$

$$= \sum_{x_r=\sum_{j=1}^{r-1} k_j+1}^{n} \sum_{x_{r-1}=\sum_{j=1}^{r-2} k_j+1}^{x_r-k_{r-1}} \cdots$$

$$\cdots \sum_{x_5=\sum_{j=1}^{4} k_j+1}^{x_6-k_5} \sum_{j=0}^{x_5-\sum_{i=1}^{4} k_i-1} \binom{j+3}{j}$$

$$= \ldots = \binom{n+r-1-\sum_{j=1}^{r-1} k_j}{r}. \qquad \text{(eq. 3)}$$

The correctness of eq. 3 can easily be verified by induction. Therefore, if $\sum_{j=1}^{r-1} k_j \leq n-1$, we have:

$$P\left(X_1 \leq X_2 - k_1, X_2 \leq X_3 - k_2, \ldots, X_i \leq X_{i+1} - k_i, \ldots \right.$$
$$\left. \ldots, X_{r-1} \leq X_r - k_{r-1}\right) = \frac{\binom{n+r-1-\sum_{j=1}^{r-1} k_j}{r}}{n^r}.$$

$\square$

Note: If our experiment is modified such that the cards are selected *without* replacement, the number of outcomes can also be determined by eq. 3. The experiment of selecting r objects from n objects (*without* replacement) is tantamount to our experiment of selecting r cards from n cards such that $k_i = 1$ for $i = 1, 2, \ldots, r-1$,

and the experiment of selecting r objects from n distinct objects (*with* replacement) is tantamount to our experiment of selecting r cards from n cards such that $k_i = 0$ for $i = 1, 2, \ldots, r-1$. This is a special case of eq. 3 and the number of outcomes is given by $\binom{n+r-1}{r}$ (see 2).

Example 1 If 3 cards are selected, with replacement, from 100 cards, numbered from 1 to 100, compute the probability that the number of the first card is at least 10 less than the number of the second card and the number of the second card is at least 20 less than the third.

Applying Theorem 1, we have: $n = 100$, $r = 3$, $k_1 = 10$, $k_2 = 20$, and

$$P\left(X_1 \le X_2 - 10,\ X_2 \le X_3 - 20\right) = \frac{\binom{100+3-1-(10+20)}{3}}{100^3}$$

$= \frac{59640}{100^3} = 0.06.$

Next, we consider the multiplicative analog of this example.

Problem 1 If r cards are selected, with replacement, from n cards, numbered from 1 to n, compute the number of ways in which the number of the i^{th} card selected is less than or equal to k_i times the number of the $(i+1)^{\text{st}}$ card.

In this case, we have, analogous to Theorem 1:

$$X_1 \le k_1 X_2,\ X_2 \le k_2 X_3, \ldots, X_i \le k_i X_{i+1}, \ldots, \le X_{r-1} \le k_{r-1} X_r.$$

Given the relations above, we can use sigma notation to determine the number of outcomes. But since determining the bounds of sigma notation is dependent on the values of n and r, it is impossible to reach a general formula. Therefore, we present more explanations in a special case by means of an example.

Example 2 Given the scenario of Example 1, suppose we want to determine the number of ways that the three cards can be selected, such that $x_1 \le 5x_2$ and $x_2 \ge 2x_3^3 + 7$. We have:

$$\begin{array}{rrl} x_3 = 1, & 9 \le x_2 \le 20, & 1 \le x_1 \le 5x_2 \\ x_3 = 1, & 21 \le x_2 \le 100, & 1 \le x_1 \le 100 \\ 2 \le x_3 \le 3, & 2x_3^3 + 7 \le x_2 \le 100, & 1 \le x_1 \le 100 \end{array}$$

So, the number of possible outcomes will be as follows:

$$\sum_{x_2=9}^{20} \sum_{x_1=1}^{5x_2} 1 + \sum_{x_2=21}^{100} \sum_{x_1=1}^{100} 1 + \sum_{x_3=2}^{3} \sum_{x_2=2x_3^3+7}^{100} \sum_{x_1=1}^{100} 1 = 20670.$$

Part II - An Application of Sigma Notation in Evaluating the Number of Non-negative Integer Solutions of an Indeterminate Equation.

Introduction

Consider the indeterminate equation in m variables:

$$\sum_{i=1}^{m} a_i x_i = n;\ m, n \in \mathbf{N}. \qquad \text{(eq. 4)}$$

For evaluating the number of solutions of eq. 4, there is a lengthy and tiresome classical method which uses generating functions. In this paper we discuss a method which is shorter than the classical method and depends mainly on the properties of sigma notation.

We begin our discussion by considering linear equations in the special case in which $a_i | a_{i+1}$ for $i = 1, 2, \ldots m-1$. Next, we consider the case of non-linear equations. Finally, we consider a method for approximating the number of non-negative integer solutions of eq. 4.

Preliminaries

We denote the set $\{0, 1, 2, \ldots\}$ by $\mathbf{N}^*$, the set of natural numbers by $\mathbf{N}$, and the Cartesian product $\prod_m \mathbf{N}$ by $\mathbf{N}^m$. We denote the set of non-negative integer solutions of $\sum_{i=1}^{m} a_i x_i = n$ by $A_{\sum_{i=1}^{m} a_i x_i = n}$, its cardinality by $\left| A_{\sum_{i=1}^{m} a_i x_i = n} \right|$, and the integer part of α by $[\alpha]$.

Method for Linear Equations

We devote this section to finding the number of solutions of linear equations in m variables in $\mathbf{N}^{*^m}$.

Lemma 1 Suppose that for $i = 1, 2, 3, \ldots, m-1$, $a_i | a_{i+1}$ and that $a_m | n$. If

$$S = \left\{ (x_1, x_2, \ldots, x_m) : 0 \le x_1 \le \frac{n - \sum_{i=2}^{m} a_i x_i}{a_1}, \right.$$

$$0 \le x_2 \le \frac{n - \sum_{i=3}^{m} a_i x_i}{a_2}, \ldots, 0 \le x_{m-1} \le \frac{n - a_m x_m}{a_{m-1}},$$

$$\left. 0 \le x_m \le \frac{n}{a_m};\ x_i \in \mathbf{N}^* \text{ for } i = 1, 2, \ldots m \right\}.$$

Then we have:

$$\left|A_{\sum_{i=1}^{m} a_i x_i \le n}\right| = |S| = \sum_{x_m=0}^{\frac{n}{a_m}} \sum_{x_{m-1}=0}^{\frac{n-a_m x_m}{a_{m-1}}} \cdots \sum_{x_2=0}^{\frac{n-\sum_{i=3}^{m} a_i x_i}{a_2}} \sum_{x_1=0}^{\frac{n-\sum_{i=2}^{m} a_i x_i}{a_1}} 1.$$

Example 3 Calculate the number of nonnegative integer solutions of $3x + 12y + 60z \le 3000$.

By Lemma 1we have:

$$|A_{3x+12y+60z\le 3000}| = \sum_{z=0}^{\frac{3000}{60}} \sum_{y=0}^{\frac{3000-60z}{12}} \sum_{x=0}^{\frac{3000-12y-60z}{3}} 1 = 2165426.$$

Now suppose the condition $a_m|n$ does not hold. In this case, slight changes in Lemma 1 lead us to the following proposition:

Proposition 1 Suppose that for $i = 1, 2, 3, \ldots, m-1$, $a_i|a_{i+1}$. Then

$$\left|A_{\sum_{i=1}^{m} a_i x_i \le n}\right| = |S| = \sum_{x_m=0}^{\left[\frac{n}{a_m}\right]} \sum_{x_{m-1}=0}^{\left[\frac{n}{a_{m-1}}\right]-\frac{a_m x_m}{a_{m-1}}} \cdots$$

$$\cdots \sum_{x_2=0}^{\left[\frac{n}{a_2}\right]-\frac{\sum_{i=3}^{m} a_i x_i}{a_2}} \sum_{x_1=0}^{\left[\frac{n}{a_1}\right]-\frac{\sum_{i=2}^{m} a_i x_i}{a_1}} 1.$$

Remark: A special case of the previous proposition ($m = 2$) yields a simple version of the formula. We give it here.

$$|A_{a_1x_1+a_2x_2\le n}| = \left(\left[\frac{n}{a_2}\right]+1\right)\left(\left[\frac{n}{a_1}\right]+1-\frac{a_2}{2a_1}\left[\frac{n}{a_2}\right]\right).$$

Editor's Note: Lemma 1 and Proposition 1 are analogs of Proposition 6.1 in [2], which deals with the non-negative integer solutions of linear equations. The proofs of Lemma 1 and Proposition 1 follow directly from the statement of Proposition 6.1 in [2] and the arguments contained in Theorem 1.

We will now illustrate the previous proposition by a concrete example.

Example 4

$$|A_{3x+12y+60z\leq 2999}| = \sum_{z=0}^{49}\sum_{y=0}^{249-5z}\sum_{x=0}^{999-20z-4y} 1 = 2159000.$$

Adding a Non-linear Twist:

By Proposition 1, we can evaluate the number of solutions of $\sum_{i=1}^{m-1} a_i x_i + a_m (x_m)^k = n$, where $k, m, n \in \mathbf{N}$, and such that $a_i | a_{i+1}$.

To find non-negative integer solutions of $\sum_{i=1}^{m-1} a_i x_i + a_m (x_m)^k \leq n$, given the same hypotheses, we proceed as in the linear case. First we must determine the range of $x_1, x_2, \ldots, x_m$. Given $\sum_{i=1}^{m-1} a_i x_i + a_m (x_m)^k = n$, it follows that:

$$0 \leq x_1 \leq \left[\frac{n}{a_1}\right] - \frac{\sum_{i=2}^{m-1} a_i x_i + a_m (x_m)^k}{a_1}$$

$$0 \leq x_2 \leq \left[\frac{n}{a_2}\right] - \frac{\sum_{i=3}^{m-1} a_i x_i + a_m (x_m)^k}{a_2}$$

$$\vdots$$

$$0 \leq x_{m-1} \leq \left[\frac{n}{a_{m-1}}\right] - \frac{a_m (x_m)^k}{a_{m-1}}$$

$$0 \leq x_m \leq \left[\sqrt[k]{\frac{n}{a_m}}\right].$$

Consequently,

$$\left|A_{\sum_{i=1}^{m-1} a_i x_i + a_m (x_m)^k \leq n}\right| = \sum_{x_m=0}^{\left[\sqrt[k]{\frac{n}{a_m}}\right]} \sum_{x_{m-1}=0}^{\left[\frac{n}{a_{m-1}}\right] - \frac{a_m (x_m)^k}{a_{m-1}}} \cdots$$

$$\cdots \sum_{x_2=0}^{\left[\frac{n}{a_2}\right] - \frac{\sum_{i=3}^{m-1} a_i x_i + a_m (x_m)^k}{a_2}} \sum_{x_1=0}^{\left[\frac{n}{a_1}\right] - \frac{\sum_{i=2}^{m-1} a_i x_i + a_m (x_m)^k}{a_1}} 1.$$

Example 5 Find the number of solutions in $\mathbf{N}^{*^3}$ of $x + 3y + 9z^4 \leq 254$ and $x + 3y + 9z^4 = 254$.

Since our solutions are in $\mathbf{N}^{*^3}$, we deduce that:

$$\begin{cases} 0 \le x \le 254 - 3y - 9z^4 \\ 0 \le y \le \lfloor \frac{254}{3} \rfloor - 3z^4 \\ 0 \le z \le \left\lfloor \sqrt[4]{\frac{254}{9}} \right\rfloor \end{cases} \Rightarrow \begin{cases} 0 \le x \le 254 - 3y - 9z^4 \\ 0 \le y \le 84 - 3z^4 \\ 0 \le z \le 2 \end{cases}$$

$$\Rightarrow \left|A_{x+3y+9z^4 \le 254}\right| = \sum_{z=0}^{2} \sum_{y=0}^{84-3z^4} \sum_{x=0}^{254-3y-9z^4} 1 = 23283.$$

Similarly, we have:

$$\left|A_{x+3y+9z^4 \le 253}\right| = \sum_{z=0}^{2} \sum_{y=0}^{84-3z^4} \sum_{x=0}^{253-3y-9z^4} 1 = 23079.$$

Therefore, it follows that:

$$\begin{aligned} \left|A_{x+3y+9z^4=254}\right| &= \left|A_{x+3y+9z^4 \le 254}\right| - \left|A_{x+3y+9z^4 \le 253}\right| \\ &= 23283 - 23079 = 204. \end{aligned}$$

Corollary 1 Suppose that $k, m, n \in \mathbf{N}^*$. Then

$$\left|A_{x_1+x_2+\ldots+x_m+a(x_{m+1})^k=n}\right| = \sum_{x=0}^{\lfloor \sqrt[k]{\frac{n}{x}} \rfloor} \binom{n - ax^k + m - 1}{m-1}.$$

Our method can be modified for application to problems in which there are constraints on the variables. We consider such an example.

Example 6 Suppose that we want to determine the number of solutions in $\mathbf{N}^{*^3}$ of $x + 6y + 12z^3 \le 20177$, subject to the following constraints:

$$25 \le x \le 400, \quad 17 \le y \le 60, \quad 10 \le z \le 32.$$

Given that $x+6y+12z^3 \le 20177$, we deduce that $10 \le z \le 11$. Neither this further constraint on z nor the constraints inherent in the given inequality pose any further restrictions on the variables x and y. So the number of solutions in $\mathbf{N}^{*^3}$, given the constraints, is:

$$\sum_{z=10}^{11} \sum_{y=17}^{60} \sum_{x=25}^{400} 1 = 33088$$

Method of Approximating the Number of Solutions

The method employed in the previous section is more appropriate than the generating function method and is sufficient for our needs. Nevertheless, in this section, we present some other methods for finding the number of approximate solutions.

The area of the region bounded by the line $a_1x_1 + a_2x_2 = n$; $a_1, a_2, n > 0$ and the coordinate axes is equal to $\frac{\left(\frac{n}{a_1}\right)\left(\frac{n}{a_2}\right)}{2}$. Similarly the volume of the solid bounded by the plane $a_1x_1 + a_2x_2 + a_3x_3 = n$; $a_1, a_2, a_3, n > 0$ and the coordinate planes is equal to $\frac{\left(\frac{n}{a_1}\right)\left(\frac{n}{a_2}\right)\left(\frac{n}{a_3}\right)}{6}$. Therefore, the approximate number of lattice points in these regions (or equivalently, the approximate number of non-negative integer solutions of $a_1x_1+a_2x_2 \le n$ and $a_1x_1+a_2x_2+a_3x_3 \le n$) is given by $\frac{\left(\frac{n}{a_1}+1\right)\left(\frac{n}{a_2}+1\right)}{2}$ and $\frac{\left(\frac{n}{a_1}+1\right)\left(\frac{n}{a_2}+1\right)\left(\frac{n}{a_3}+1\right)}{6}$ respectively.

By extending the two and three variable cases, the approximate number of non-negative integer solutions of can be written as:

$$\left|A_{\sum_{i=1}^{m} a_i x_i \le n}\right| \approx \frac{\prod_{i=1}^{m}\left(\frac{n}{a_i}+1\right)}{m!}, \qquad \text{(eq. 5)}$$

This approximation can be improved as follows:

$$\left|A_{\sum_{i=1}^{m} a_i x_i \le n}\right| \approx \frac{\prod_{i=1}^{m}\left(\frac{n}{a_i}+1\right)}{m!} + \left(1-\frac{1}{m!}\right)\left(\frac{\prod_{i=1}^{m}\left(\frac{n}{a_i}+1\right)}{m!} - \frac{\prod_{i=1}^{m}\left(\frac{n-1}{a_i}+1\right)}{m!}\right). \qquad \text{(eq. 6)}$$

Example 7 Consider the inequality $7x_1 + 66x_2 \le 10000$. Then eq. 6 implies that

$$\begin{aligned} \left|A_{7x_1+66x_2\le 10000}\right| &\approx \frac{\left(\frac{10000}{7}+1\right)\left(\frac{10000}{66}+1\right)}{2} \\ &+ \left(\frac{1}{2}\right)\left(\frac{\left(\frac{10000}{7}+1\right)\left(\frac{10000}{66}+1\right)-\left(\frac{9999}{7}+1\right)\left(\frac{9999}{66}+1\right)}{2}\right) \\ &= 109026.51 \end{aligned}$$

Let Δ denote the difference between the exact number of solutions of the inequality and the approximate number of solutions as given above. Then

$$\Delta = 109027 - 109026.51 \approx 0.49$$

By extending Lemma 1, the following expression satisfies every indeterminate inequality.

$$\left|A_{\sum_{i=1}^{m} a_i x_i \leq n}\right| = \sum_{x_m=0}^{\left[\frac{n}{a_m}\right]} \sum_{x_{m-1}=0}^{\left[\frac{n-a_m x_m}{a_{m-1}}\right]} \cdots \sum_{x_2=0}^{\left[\frac{n-\sum_{i=3}^{m} a_i x_i}{a_2}\right]} \sum_{x_1=0}^{\left[\frac{n-\sum_{i=2}^{m} a_i x_i}{a_1}\right]} 1.$$

(eq. 8)

The number of solutions in $\mathbf{N}^{*^m}$ of the corresponding indeterminate equation can be computed using eq. 8 as follows:

$$\left|A_{\sum_{i=1}^{m} a_i x_i = n}\right| = \sum_{x_m=0}^{\left[\frac{n}{a_m}\right]} \sum_{x_{m-1}=0}^{\left[\frac{n-a_m x_m}{a_{m-1}}\right]} \cdots \sum_{x_2=0}^{\left[\frac{n-\sum_{i=3}^{m} a_i x_i}{a_2}\right]} \sum_{x_1=0}^{\left[\frac{n-\sum_{i=2}^{m} a_i x_i}{a_1}\right]} 1$$

$$- \sum_{x_m=0}^{\left[\frac{n-1}{a_m}\right]} \sum_{x_{m-1}=0}^{\left[\frac{n-1-a_m x_m}{a_{m-1}}\right]} \cdots \sum_{x_2=0}^{\left[\frac{n-1-\sum_{i=3}^{m} a_i x_i}{a_2}\right]} \sum_{x_1=0}^{\left[\frac{n-1-\sum_{i=2}^{m} a_i x_i}{a_1}\right]} 1$$

$$\approx \int_{-\frac{1}{2}}^{\frac{n}{a_m}} \int_{-\frac{1}{2}}^{\frac{n-a_m x_m}{a_{m-1}}} \cdots \int_{-\frac{1}{2}}^{\frac{n-\sum_{i=3}^{m} a_i x_i}{a_2}} \int_{-\frac{1}{2}}^{\frac{n-\sum_{i=2}^{m} a_i x_i}{a_1}} dx_1 dx_2 \ldots dx_m$$

$$- \int_{-\frac{1}{2}}^{\frac{n-1}{a_m}} \int_{-\frac{1}{2}}^{\frac{n-1-a_m x_m}{a_{m-1}}} \cdots \int_{-\frac{1}{2}}^{\frac{n-1-\sum_{i=3}^{m} a_i x_i}{a_2}} \int_{-\frac{1}{2}}^{\frac{n-1-\sum_{i=2}^{m} a_i x_i}{a_1}} dx_1 dx_2 \ldots dx_m.$$

Clearly, evaluating the iterated integrals in the above expression is simpler than computing the corresponding sums. However, as the number of integrals increases, computation of the iterated integrals will also be difficult. Therefore, we try to find a simple closed form. First, consider the two variable case.

$$\begin{aligned} \left|A_{a_1 x_1 + a_2 x_2 \leq n}\right| &\approx \int_{-\frac{1}{2}}^{\frac{n}{a_2}} \int_{-\frac{1}{2}}^{\frac{n-a_2 x_2}{a_1}} dx_1 dx_2 \\ &= \frac{n}{2a_1 a_2}\left(n + (a_1 + a_2)\right) + \left(\frac{a_2}{8a_1} + \frac{1}{4}\right). \quad \text{(eq. 9)} \end{aligned}$$

Let $G(n)$ denote the value of eq. 9 as a function of n. To determine the number of solutions of $a_1x_1 + a_2x_2 = n$, we should compute the difference $G(n) - G(n-1)$. For this purpose, we can ignore the constant $\left(\frac{a_2}{8a_1} + \frac{1}{4}\right)$. In other words, we have:

$$|A_{a_1x_1+a_2x_2=n}| \approx G(n) - G(n-1) : G(n) = \frac{n}{2a_1a_2}(n + (a_1 + a_2)).$$

In general, we get the following formula:

$$\left|A_{\sum_{i=1}^m a_ix_i=n}\right| \approx G(n) - G(n-1);$$

$$\text{where } G(n) = \frac{\left(n+\frac{\sum_{i=3}^m a_i}{2}\right)^{m-1}}{m!\prod_{i=1}^m a_i}\left(\left(n + \frac{\sum_{i=3}^m a_i}{2}\right) + \frac{m}{2}(a_1 + a_2)\right).$$

(eq. 10)

Eq. 10 can be proved by induction.

Example 8 We conclude by presenting tables comparing the approximate number of solutions of indeterminate inequalities to the exact number of solutions, which we have obtained via a computer program. The examples given show that the accuracy of eq. 10 decreases as n and the number of variables increase. However, when the coefficients of variables, n, and $n-1$ are pair-wise coprime, the aforementioned method is more accurate.

$\sum_{i=1}^m a_ix_i \le n$	eq. 10	$\left\|A_{\sum_{i=1}^m a_ix_i \le n}\right\|$
$6x_1 + 18x_2 \le 100$	58.02	57
$6x_1 + 18x_2 \le 103$	61.17	63
$6x_1 + 17x_2 \le 100$	60.88	61
$6x_1 + 17x_2 \le 103$	64.21	65
$6x_1 + 18x_2 \le 1006$	4797.75	4788
$6x_1 + 18x_2 \le 1009$	4826.13	4845
$6x_1 + 17x_2 \le 1001$	5025.25	5030
$6x_1 + 17x_2 \le 1004$	5055	5060
$6x_1 + 18x_2 + 36x_3 \le 1042$	52807	52635
$6x_1 + 18x_2 + 36x_3 \le 1045$	53251	53535
$6x_1 + 19x_2 + 36x_3 \le 1042$	50097	50148
$6x_1 + 19x_2 + 36x_3 \le 1045$	50520	50579
$6x_1 + 19x_2 + 37x_3 \le 1071$	52878	52931
$6x_1 + 19x_2 + 37x_3 \le 1074$	53311	53365
$6x_1+19x_2+35x_3+43x_4 \le 1700$	2166912	2168130
$6x_1+19x_2+37x_3+43x_4+49x_5 \le 1700$	16613380	16611400
$6x_1+19x_2+37x_3+43x_4+49x_5 \le 1666$	15084030	15081360

$\sum_{i=1}^{m} a_i x_i \leq n$	Δ	$\delta = \frac{\Delta}{\lvert A_{\sum_{i=1}^{m} a_i x_i \leq n} \rvert}$
$6x_1 + 18x_2 \leq 100$	1.02	0.018
$6x_1 + 18x_2 \leq 103$	1.83	0.029
$6x_1 + 17x_2 \leq 100$	0.12	0.002
$6x_1 + 17x_2 \leq 103$	0.79	0.012
$6x_1 + 18x_2 \leq 1006$	9.75	0.002
$6x_1 + 18x_2 \leq 1009$	18.87	0.004
$6x_1 + 17x_2 \leq 1001$	4.75	0.001
$6x_1 + 17x_2 \leq 1004$	5	0.001
$6x_1 + 18x_2 + 36x_3 \leq 1042$	172	0.003
$6x_1 + 18x_2 + 36x_3 \leq 1045$	284	0.005
$6x_1 + 19x_2 + 36x_3 \leq 1042$	51	0.001
$6x_1 + 19x_2 + 36x_3 \leq 1045$	59	0.001
$6x_1 + 19x_2 + 37x_3 \leq 1071$	53	0.001
$6x_1 + 19x_2 + 37x_3 \leq 1074$	54	0.001
$6x_1+19x_2+35x_3+43x_4 \leq 1700$	1218	0.001
$6x_1+19x_2+37x_3+43x_4+49x_5 \leq 1700$	1980	0.000
$6x_1+19x_2+37x_3+43x_4+49x_5 \leq 1666$	2670	0.000

References

[1] M.H. Poursaeed, *The Presentation and Generalization of a Method to a Solution to a Probability Problem,* Gulchin-Iriyazi, Journal of Mathematical Statistics,Vol.1, Spring, 1996.

[2] Sheldon Ross, *A First Course In Probability,* MacMillan Publishing Co. 1976.

Department of Mathematics
University of Lorestan
Khorramabad, Iran
P.O.Box 68135-465
Fax: +98 661 22782
poursaeed2001@yahoo.com